APPENDIX 2

1 - Orthogonal filter banks with two sub-bands

A filter bank with two sub-bands can be shown according to the diagram of Figure 3. In order to meet the condition of non-aliasing, the synthesis filters $G_0(z)$ and $G_1(z)$ are directly deduced from the analysis filters by $G_0(z) = H_1(-z)$ and $G_1(z) = -H_0(-z)$.

It can then be shown [4] that the condition of perfect reconstruction (PR) can be expressed solely from the polyphase matrix $H_{[H_0,H_1]}$ (z) of the analysis bank:

$$\mathrm{H}_{[H_0,H_1]}(z) = \begin{bmatrix} H_{0,0}(z) & H_{0,1}(z) \\ H_{1,0}(z) & H_{1,1}(z) \end{bmatrix}, \tag{9}$$

where the values of $H_{k,l}(z)$ are the polyphase components for each filter $H_k(z)$, namely:

$$H_k(z) = \sum_{l=0}^{1} z^{-l} H_{k,l}(z^2) \tag{10}$$

In the case of the orthogonal banks with two sub-bands, the filters are 2n+1 odd-order filters and the polyphase components are all of the same n degree. The bank is then PR if the determinant of the polyphase matrix is a monome. In other words if:

$$\mathrm{Det}\ \mathrm{H}_{[H_0,H_1]}(z) = H_{0,0}(z)H_{1,1}(z) - H_{1,0}(z)H_{0,1}(z) = \beta z^{-n}\ ; \tag{11}$$

with β as a non-null constant.

Let $\hat{Q}$ (z) be the mirror filter of the mth order filter $Q(s)$, namely: $\hat{Q}$ (s) = $z^{-m}Q(z^{-1})$. The orthogonal bank is characterized by filters $H_0(z)$ and $H_1(z)$ said to be quadrature conjugates, which can be written as follows $H_1(z) = \pm z^{-(2n+1)}H_0(z^{-1})$. Hereinafter, unlike the presentation given in [4], we choose the writing with a plus sign. Thus, the polyphase components meet the following relationship:

$$H_{0,1}(z) = -\hat{H}_{1,0}(z)\ ,\quad H_{1,1}(z) = \hat{H}_{0,0}(z)\ . \tag{12}$$

Consequently, the polyphase matrix can be written as follows:

$$\mathrm{H}_{[H_0,H_1]}(z) = \begin{bmatrix} H_{0,0}(z) & -\hat{H}_{1,0}(z) \\ H_{1,0}(z) & \hat{H}_{0,0}(z) \end{bmatrix}, \tag{13}$$

and the relationship (11) can be written in the form:

$$\text{Det } \mathrm{H}_{[H_0,H_1]}(z) = H_{0,0}(z)\hat{H}_{0,0}(z) + H_{1,0}(z)\hat{H}_{1,0}(z) = \beta z^{-n} , \qquad (14)$$

The output signal is then such that $\hat{X}\ (s) = \gamma z^{-(2n+1)}X(z)$ with γ as a non-null constant. The matrix $H_{[H_0,H_1]}$ (z) is para-unitary and can be cascaded with [4]:

$$\mathrm{H}_{[H_0,H_1]}(z) = g\mathrm{A}(\alpha_n)\Lambda(z)\mathrm{A}(\alpha_{n-1})\ldots\Lambda(z)\mathrm{A}(\alpha_0), \qquad (15)$$

where g is a non-null standardization constant, α_0, ..., α_n are n+1 real numbers and where the matrices A(α) for α real and Λ(z) are defined by:

$$\mathrm{A}(\alpha) = \begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix} , \quad \Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \qquad (16)$$

An embodiment of a synthesis bank based on this mathematical writing produces the lattice depiction of Figure 4.

2. General characteristics of the case of the linear phase matched pair for M=4

For the linear phase matched pair, we necessarily have $F_T(z)$ and $F_R(z)$ symmetrical such that $F_T(z) = F_R(z)$, which can be written with only one expression $F(z)$. Let N be the order of $F(z)$. We therefore have:

$$F(z) = \sum_{k=0}^{N} f_k z^{-k} \qquad (17)$$

The produced filter $P(z)$ is then expressed by:

$$P(z) = F^2(z) = \sum_{k=0}^{2N} p(k) z^{-k} \qquad (18)$$

with of course $p(k) = p(2N - k)$. $P(z)$ meets the zero ISI condition (5) if:

$$p(k) = 0 \text{ for } k - N = 4l,\ l \neq 0 \qquad (19)$$

It is possible to make a first observation with regard to the $N = 4n$ order filters.

Proposition 1 - *Let $F(z)$ be a $4n$ order symmetrical filter. $F(z)$ cannot be a null ISI filter.*

In order that $F(z)$, a symmetrical filter, may be a null ISI filter, we must have $f_0 \neq 0$. The highest degree monome of $P(z)$ is f_0^2 z^{-8n} while its 4n degree central term is z^{-1}. Since the difference in degrees is a multiple of 4, $F(z)$ is not at null ISI.

The polyphase decomposition of $F(z)$ for $M = 4$ can be written in the following form:

$$F(z) = F_0(z^4) + z^{-1}F_1(z^4) + z^{-2}F_2(z^4) + z^{-3}F_3(z^4)\,. \tag{20}$$

with

$$F_l(z) = \sum_{n=0}^{n_l} f_{4n+l} z^{-n} \tag{21}$$

To carry out the analysis of the different possible cases as a function of the Nth order, the property of symmetry of $F(z)$ and the fact that $4n_l + l \leq N$ will be used.

• If $N = 4n$, the degree of $F_0(z)$ is equal to n and that of the values $F_i(z)$, $i = 1, 2, 3$ is strictly below n. According to Proposition 1, it is known that this case is of no use whatsoever for our problem.

• If $N = 4n + 1$, $F_1(z)$ is an n degree value, $F_0(z)$ is a $\leq n$ degree value while $F_2(z)$ and $F_3(z)$ are $\leq n - 1$ degree values. $F_1(z)$ then corresponds by mirror symmetry to $F_0(z)$. $F_1(z)$ is therefore actually an n degree value. We therefore have:

$$F_0(z) = \hat{F}_1(z) \text{ et } F_3(z) = \hat{F}_2(z) \tag{22}$$

• If $N = 4n + 2$, we get $n_2 = n$, $n_1 = n$ and $n_3 = n - 1$. $F_2(z)$ is symmetrical to $F_0(z)$. We therefore have:

$$F_2(z) = \hat{F}_0(z) \;\;,\;\; F_1(z) = \hat{F}_1(z) \;\;,\;\; F_3(z) = \hat{F}_3(z) \tag{23}$$

• If $N = 4n + 3$, $4n_l + 1 \leq 4n + 3$, $\forall l$ $(0 \leq l \leq 3)$, and the filters F_i are all $\leq n$ degree

filters. Indeed, the highest-degree term of $F_0(z)$ is the $4n + 3$ degree term giving an n degree term at $F_3(z)$. The highest-degree term of $F_3(z)$ corresponds by symmetry to the constant term of $F_0(z)$. We therefore have:

$$F_3(z) = \hat{F}_0(z) \text{ et } F_2(z) = \hat{F}_1(z) \tag{24}$$

Note: In the equation (22), we have deliberately written $F_0(z) = \hat{F}_1(z)$ and not $F_1(z) = \hat{F}_0(z)$ because it is certain that the degree of $F_1(z)$ is n since the constant term of $F_0(z)$ is not zero but not that the degree of $F_0(z)$ is n. Furthermore, this case where $f_1 = f_{N-1} = 0$ is considered further below.

The product $F^2(z)$ is then developed in taking account of the decomposition (20) and the following characterization of the filters $F(z)$ verifying the null ISI property is obtained.

Theorem 1 - *Let $F(z)$ be an Nth order symmetrical filter verifying $F(z = 0) \neq 0$. Then, depending on the values of N, $F(z)$ is at null ISI if and only if:*

- $N = 4n + 1$,

$$F_1(z)\hat{F}_1(z) + z^{-1}F_2(z)\hat{F}_2(z) = \gamma z^{-n}, \tag{25}$$

- If $N = 4n + 2$,

$$2F_0(z)\hat{F}_0(z) + F_1^2(z) + z^{-1}F_3^2(z) = \gamma z^{-n}, \tag{26}$$

- If $N = 4n + 3$,

$$F_0(z)\hat{F}_0(z) + F_1(z)\hat{F}_1(z) = \gamma z^{-n}, \tag{27}$$

where γ is a non-null constant.

The following theorem shows that all the $N = 4(n - 1) + 3$ order solutions are obtained from $N = 4n + 1$ order solutions.

Theorem 2 - *Let $F(z)$ be a null ISI symmetrical filter of the $4n + 1$, $n \geq 1$ order. Then if $F_i(z)$, $i = 0, ..., 3$ designates the polyphase components, $F_0(z)$ is of a degree strictly*

below n and $F_1(z)$ can be written in the following form: $F_1(z) = z^{-1}K_1(z)$. We then have $\hat{K}_1(z) = F_0(z)$ and the filter $\overline{F}(z)$ with polyphase components $[F_0(z), F_2(z), F_3(z), K_1(z)]$ is a symmetrical $4(n-1)+3$ order, null ISI filter.

Demonstration - This immediately results from the relationship (25) since the constant term of the member on the left is null. We therefore have

$F_1(z=0)\hat{F}_1(z=0) = f_0 f_1 = 0$. Since $F_1(z)$ contributes with f_0 to the highest-degree term, $f_0 \neq 0$ and therefore $f_1 = 0$. Writing $F_1(z)$ in the form $F_1(z) = z^{-1}K_1(z)$ and knowing that $\hat{F}_1(z) = \hat{K}_1(z)$, the relationship (26) can also be written as follows:

$$z^{-1}K_1(z)\hat{K}_1(z) + z^{-1}F_2(z)\hat{F}_2(z) = \gamma z^{-n} , \qquad (28)$$

Since $\hat{K}_1(z) = F_0(z)$ and therefore $K_1(z) = \hat{F}_0(z)$, we obtain the relationship (27) for a filter where $K_1(z)$ is substituted for $F_3(z)$ and therefore has the polyphase components: $[F_0(z), F_2(z), F_3(z), K_1(z)]$. It is therefore possible, in one example, to verify the passage from a 9th order filter to a 7th order filter.

Example

$$\begin{aligned} F_0(z) &= f_0 + f_4 z^{-1} + f_1 z^{-2} &&= f_0 + f_4 z^{-1} \\ F_1(z) &= f_1 + f_4 z^{-1} + f_0 z^{-2} &&= z^{-1}(f_4 + f_0 z^{-1}) \quad = z^{-1}K_1(z) \\ F_2(z) &= f_2 + f_3 z^{-1} \\ F_3(z) &= f_3 + f_2 z^{-1} \end{aligned} \qquad (29)$$

The composite filter of $[F_0(z), F_2(z), F_3(z), K_1(z)]$ is a 7th order filter and is then expressed by:

$$f_0 + f_2 z^{-1} + f_3 z^{-2} + f_4 z^{-3} + f_4 z^{-4} + f_3 z^{-5} + f_2 z^{-6} + f_0 z^{-7} \qquad (30)$$

A third theorem shows that the $N = 4n + 3$ order solutions are obtained from the orthogonal two sub-band filter bands.

Theorem 3 - *Let $[H_0(z), H_1(z)]$ be a bank of $2n+1$ order orthogonal filters with two sub-bands whose polyphase matrix is given by (13). Then the filter $\bar{H}(z)$ for which the four polyphase components are $[H_{0,0}(z), H_{1,0}(z), \hat{H}_{1,0}(z), \hat{H}_{0,0}(z)]$ is a linear phase, $4n+3$ order, null ISI filter. Reciprocally, all the filters of this type are obtained from orthogonal filter banks with two sub-bands.*

Demonstration - The demonstration is immediate since the $4n + 3$ order linear phase, null ISI filters are characterized by the relationship (27) which, in the terms of the theorem, becomes equivalent to (14).